Moore-Penrose Generalized Inverse to Kronecker Product Matrix Boundary Value Problems

Appa Rao B.V, T.S.Rao

Abstract: This paper deals with the existence-uniqueness to Kronecker product (KP) 3- point boundary value problems(TBVP) for the first order linear system as well as nonlinear matrix differential equations with the help of Moore-Penrose generalized inverse and Banach fixed point theory.

Keywords: Existence-uniqueness, Moore-Penrose generalized inverse, Rectangular matrices.

I. INTRODUCTION

The importance of system of differential equations and their occurrence in many physical problems are well known. Many authors[[4]-[6]&[11]] have studied the problems of existence-uniqueness to two and 3- point boundary value problems(BVP's) for the first order matrix system of differential equations.

Here we focus our attention to KPTBVP associated with the system

$$(A(t) \otimes B(t))x'(t) + (C(t) \otimes D(t))x(t) = g(t),$$
(1.1)

where $A(t), B(t)) \in C^2[a,c]$ and $C(t), D(t)) \in C^1[a,c]$. A, C are rectangular matrices of order $m \times n$. B, D are rectangular matrices of order $p \times q$ and g(t) is a column mp- vector whose components are in C1[a, c], satisfying

$$(L_1 \otimes M_1)x(a) + (L_2 \otimes M_2)x(b) + (L_3 \otimes M_3)x(c) = \beta,$$
(1.2)

where Lj's and Mj's (j=1,2,3) are nth and qth order square matrices respectively and β is a column nq-vector.

The results of this paper are established under the assumptions, when $(A(t) \otimes B(t))$ is $mp \times nq$ matrix, $(C(t) \otimes D(t))$ is in the column space of $(A(t) \otimes B(t))$ and Lj's and Mj's (j=1,2,3) are non-singular square matrices.

Recently many authors [[2],[3],[9] and [10]] have studied the applications of generalized inverses to BVPs associated with the system of first order differential equations. The rest of the paper is is as follows: mainly in 2nd section we obtain some fundamental results on generalized inverses, KP of matrices and BVPs.

In section 3 we investigate the existence- uniqueness of solutions associated with TPBVPs (1.1), Boundary condition(B.C)(1.2). The Green's function or matrix as well as associated Green's function or matrix properties are developed with the help of uniqueness property of the Moore-Penrose generalized inverse. Further results of existence and uniqueness of solutions of KP non-linear TPBVPs are presented in section 4.

$a \le t \le c$ The situation differs when 'g' is non-linear and the boundary conditions are non-homogeneous. We obtain the criteria for existence and uniqueness to KPTBVP associated with the nonlinear system

$$(A \otimes B)x' + (C \otimes D)x = g(t,x), \qquad a \le t \le c$$
(1.3)

where A, B, C and D are stated as earlier and $g(t,x(t)) \in C([a,c] \times R^{mp})$ satisfying(1.2). These results are established under the assumption that the related homogeneous boundary value problems. $(g(t,x(t)) = 0, \beta = 0)$ has only the trivial solution. These results depend on Banach fixed point theorem and Green's matrix.

II. PRELIMINARIES

In this paper we assume that the reader is familiar with fundamental concepts of KP matrices. In this section we develop some basic results relating to generalized inverses,

Here we denote regular inverse of a matrix P by P-1 and its Moore-Penrose generalized inverse by P+

Definition 2.1 [8]. Let $P(t) \in R^{m \times n}$. The $n \times m$ matrix P$^+$ is said to be the Moore-Penrose generalized inverse of P, if each of the following is satisfied.

    (i)   PP$^+$P=P

    (ii) P$^+$PP$^+$=P$^+$

    (iii) (PP$^+$)$^+$=PP$^+$

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(iv) $(P^+P)^+ = P^+P$

Definition 2.2 A matrix B(t) is said to be column space of A(t) if there exists a square matrix K(t) such that B(t)=A(t)K(t) .

Result 2.1 Suppose the matrices C(t) and D(t) are in the column space of A(t) and B(t) respectively, then $(C(t) \otimes D(t))$ is in the column space of $(A(t) \otimes B(t))$.

Proof: Since C(t), D(t) is in the column space of the rectangular matrices A(t), B(t) respectively, then there exist square matrices $K_1(t)$, $K_2(t)$ such that C(t)=A(t)K_1(t) and D(t) = B(t) K_2(t).

Consider the Kronecker product matrix

$$(C(t) \otimes D(t)) = A(t)K_1(t)) \otimes B(t)K_2(t)$$
$$= (A(t) \otimes B(t))(K_1(t) \otimes K_2(t)).$$

Thus $(C(t) \otimes D(t))$ is in the column space of $(A(t) \otimes B(t))$.

Consider the equation

$$Py = q \qquad (2.1)$$

where $P(t) \in R^{m \times n}, m \geq n$.

Theorem 2.1. [12]

a) The following statements are equivalent:

   (i)   equation(2.1) is consistent

   (ii) $PP^+q = q$

   (iii)Rank P=rank[P;q] i.e., q is the linear combination of columns of P

b)The solution, if it exists is unique if and only if P has full rank(rank P=n). In fact the general solution is given by

$$y = P^+q + (I - P^+P)z \qquad (2.2)$$

where $z \in R^n$ is arbitrary. The unique least squares solution is y=P^+q.

Result 2.2[1] The equation

$$PY = Q \qquad (2.3)$$

is consistent if and only if $[I_m - PP^+]Q = O$, where P , Y and Q are matrices of order $m \times n$, $n \times m$ and $m \times m$ respectively.

Result 2.3[1] The equation (2.3) is consistent, then its general solution is of the form

$$Y = P^+Q + (I_n - P^+P)U$$

where U is an arbitrary matrix of order $n \times m$.

Result 2.4 [7] Any solution x(t) of the system (1.1), the non-singular KP matrix $(A(t) \otimes B(t))$ is is of the form

$$x(t) = X(t)L + X(t) \int_a^t X^{-1}(s)A^{-1}(s) \otimes B^{-1}(s))g(s)ds$$

Here the fundamental matrix X(t) of the homogeneous equation associated with (1.1) and L is a constant mp-vector.

Result 2.5 Let A^+(t), B^+(t) be the generalized inverses of A and B respectively and suppose $(C(t) \otimes D(t))$ is in the column space of $(A(t) \otimes B(t))$ with

$(C(t) \otimes D(t)) = (A(t) \otimes B(t))(K_1(t) \otimes K_2(t))$. Then the two systems

$$(A(t) \otimes B(t))x'(t) + (C(t) \otimes D(t))x(t) = 0$$
$$(2.4)$$

and

$$x' + (K_1 \otimes K_2) x = [I_{nq} - (A^+ \otimes B^+)(A \otimes B)] v$$
$$(2.5)$$

are equivalent, where v is any constant nq-vector.

Proof: Let x_1 be a solution of (2.5) then

$$x_1' + K_1 \otimes K_2)x_1 = [I_{nq} - (A^+ \otimes B^+)(A \otimes B)] v$$

for some $v \in R^{nq}$.

$$(A \otimes B) x_1' + (A \otimes B)(K_1 \otimes K_2)x_1$$
$$= (A \otimes B)[I_{nq} - (A^+ \otimes B^+)(A \otimes B)] v$$
$$= 0.$$

Hence $(A(\otimes B)x' + (C \otimes D) x = 0$. We can prove the converse in a similar manner.

Remark2.6. In view of result 2.5 the linear system (2.1) is equivalent to

$$x' + (K_1 \otimes K_2) x = h \qquad (2.6)$$

where

$$h = (A^+ \otimes B^+)g + [I_{nq} - (A^+ \otimes B^+)(A \otimes B)] v.$$

Theorem2.2 [4] The fundamental matrix X of the homogeneous equation

$$x'(t) + (K_1(t) \otimes K_2(t))x(t) = 0$$

(2.7)

Then any solution (2.6) is of the form

$$x = XL + X \int_a^t X^{-1}(s) \; h(s) \, ds$$

where L is a constant nq-vector.

Theorem 2.3 Let

$$E = (L_1 \otimes M_1)X(a) + (L_2 \otimes M_2)X(b) + (L_3 \otimes M_3)X(c)$$
(2.8)

be the characteristic matrix for the homogeneous boundary value problem (2.7) associated with

$$(L_1 \otimes M_1)\,x(a) + (L_2 \otimes M_2)\,x(b) + (L_3 \otimes M_3)\,x(c) = 0$$
(2.9)

If rank E=r, then the index of compatibility of (2.9) is nq-r.

Note 2.1 In general the solution space of (2.4) does not constitute a finite dimensional space. This leads to the failure of classical methods for finding a solution to the equation (1.1).

III. KRONECKER PRODUCT LINEAR BVP

Now we establish the existence and uniqueness of solutions to KTBVP (2.6), satisfying (1.2) under the assumption that the matrices Lj's and Mj's (j=1,2,3) are non-singular square matrices.

Theorem3.1 The fundamental matrix X(t) of (2.7), also assume that the homogeneous BVP (2.9) is incompatible. Then there- exists a unique solution x(t) to the non-homogeneous BVP (1.1), (1.2) is given by

$$x = XE^{-1}\beta + \int_a^c G(t,s)[I_{nq} - (A^+ A \otimes B^+ B]vds + \int_a^c H(t,s)g(s)ds$$
(3.1)

here E is the characteristic matrix, G(t,s) is the Green's function and the associated Green's function H(t, s) for the homogeneous BVP.

Proof: Any solution of (2.6) by using theorem 2.2 is of the form

$$x = X \, L + X \int_a^t X^{-1}(s) \; h(s) \, ds$$

where X(t) is defined earlier and L is a constant nq-vector and will be determined uniquely from the fact that x(t) must obey the B.C(1.2). Substituting the general form of x(t) in the boundary conditions(1.2), we have

$$[(L_1 \otimes M_1)X(a) + (L_2 \otimes M_2)X(b) + (L_3 \otimes M_3)X(c)\,]L$$
$$+ (L_2 \otimes M_2)X(b)$$
$$\times \int_a^b X^{-1}(s)h(s)ds + (L_3 \otimes M_3)X(c)\int_a^c X^{-1}(s)h(s)ds = \beta$$

and thus

$$L = E^{-1}\beta - E^{-1}(L_2 \otimes M_2)X(b)$$
$$\times \int_a^b X^{-1}(s)\,h(s)\,ds$$
$$- E^{-1}(L_3 \otimes M_3)X(c)\int_a^c X^{-1}(s)\,h(s)\,ds$$

where 'E' is defined earlier. Substituting 'L' in the solution form gives

$$x(t) = X(t)E^{-1}\beta + X(t)\int_a^t X^{-1}(s)\,h(s)\,ds$$
$$- X(t)E^{-1}(L_2 \otimes M_2)X(b)$$
$$\times \int_a^b X^{-1}(s)\,h(s)\,ds$$
$$- X(t)E^{-1}(L_3 \otimes M_3)X(c)\int_a^c X^{-1}(s)\,h(s)\,ds$$

now rewriting the integrals using the fact that

$$X(t) = X(t)E^{-1}E = X(t)E^{-1}[(L_1 \otimes M_1)X(a)$$
$$+ (L_2 \otimes M_2)\,X(b) + (L_3 \otimes M_3)\,X(c)]$$

and substituting the value of

$$h = (A^+ \otimes B^+)g + [I_{nq} - (A^+ \otimes B^+)(A \otimes B)]v.$$

We get

$$x = XE^{-1}\beta$$
$$+ \int_a^c G(t,s)[I_{nq} - (A^+ A \otimes B^+ B]vds + \int_a^c H(t,s)\,g(s)\,ds$$

where

$$G(t,s) = \begin{cases} X(t)E^{-1}(L_1 \otimes M_1)X(a)X^{-1}(s), \\ \qquad a < s < t \leq b < c \\ -X(t)E^{-1}[(L_2 \otimes M_2)X(b) \\ \qquad + (L_3 \otimes M_3)X(c)]X^{-1}(s), \\ \qquad a \leq t < s < b < c \\ -X(t)E^{-1}(L_3 \otimes M_3)X(c)X^{-1}(s), \\ \qquad a \leq t < b < s < c \end{cases}$$
$$\quad {}_{t \in [a,b]}$$

$$G(t,s)_{t\in[b,c]} = \begin{cases} X(t)X^{-1}(s) - X(t)E^{-1}(L_3 \otimes M_3)X(c)X^{-1}(s), \\ \qquad\qquad a < b < s < t \leq c \\ -X(t)E^{-1}(L_3 \otimes M_3)X(c)X^{-1}(s), \\ \qquad\qquad a < b \leq t < s < c \\ X(t)E^{-1}(L_1 \otimes M_1)X(a)X^{-1}(s), \\ \qquad\qquad a < s < b < t \leq c \end{cases}$$

further

$$H(t,s)_{t\in[a,b]} = \begin{cases} X(t)E^{-1}(L_1 \otimes M_1)X(a)X^{-1}(s)(A^+ \otimes B^+)(s), \\ \qquad\qquad a < s < t \leq b < c \\ -X(t)E^{-1}[(L_2 \otimes M_2)X(b) \\ +(L_3 \otimes M_3)X(c)]X^{-1}(s)(A^+ \otimes B^+)(s), \\ \qquad\qquad a \leq t < s < b < c \\ -X(t)E^{-1}(L_3 \otimes M_3)X(c)X^{-1}(s)(A^+ \otimes B^+)(s), \\ \qquad\qquad a \leq t < b < s < c \end{cases}$$

and

$$H(t,s)_{t\in[b,c]} = \begin{cases} [X(t)X^{-1}(s) - X(t)E^{-1}(L_3 \otimes M_3)X(c)X^{-1}(s)](A^+ \otimes B^+)(s), \\ \qquad\qquad a < b < s < t \leq c \\ -X(t)E^{-1}(L_3 \otimes M_3)X(c)X^{-1}(s)(A^+ \otimes B^+)(s), \\ \qquad\qquad a < b \leq t < s < c \\ X(t)E^{-1}(L_1 \otimes M_1)X(a)X^{-1}(s)(A^+ \otimes B^+)(s), \\ \qquad\qquad a < s < b < t \leq c \end{cases}$$

the incompatibility of the homogeneous BVP and by using theorem 2.2 ensures that the characteristic matrix 'E' is of full rank. Since the matrices L_j's and M_j's (j=1,2,3) are of full rank, we can easily prove the uniqueness by using similar arguments as in Cole[4] and Murty [5].

Theorem 3.2 The Green's function $G(t,s)$ has the three properties;

(i) $G(t, s)$ as a function of 't' with fixed 's' has continuous derivatives everywhere except at t=s. At the point $t = s$, $G(t, s)$ is discontinuity of magnitude is unity and is given by

$$G(s^+, s) - G(s^-, s) = I_{nq}.$$

(ii) The formal solution $G(t,s)$ of the homogeneous BVP(2.9). The Green's function not satisfies original solution because of the discontinuity at $t = s$.

(iii) The properties (i) and (ii) is unique for the Green's function $G(t, s)$.

Theorem 3.3 Associated Green's function $H(t, s)$ has the following properties;

(i) $H(t, s)$ as a function of 't' with fixed 's' has continuous derivatives everywhere except at $t = s$. At the point $t = s$, $H(t, s)$ has jump discontinuity of unit magnitude and is given by

$$H(s^+, s) - H(s^-, s) = I_{nq}$$

(ii) The formal solution $H(t, s)$ of the homogeneous BVP(2.9). $H(t, s)$ fails to be actual solution because of the discontinuity $at\ t = s$.

(iii) The properties (i) and (ii) of $H(t, s)$ satisfying is unique.

Remark 3.1 $H(t, s)$ is also unique because the Moore-Penrose generalized inverse if it exists is unique.

IV. KRONECKER PRODUCT NON-LINEAR BOUNDARY VALUE PROBLEMS

In this section we obtain existence - uniqueness of solutions of KP non-linear TPBVP (1.3), (1.4). Here the function $g(t, x)$ assume that it satisfies Lipschitz condition

$$\left| g(t,x) - g(t,z) \right| \leq M \left| x-z \right|, \forall\ (t,x),(t,z) \in [a,c] \times R^{nq \times nq},$$

$$(4.1)$$

here M is a nonnegative constant and $\left| . \right|$ denotes the usual norm as in the text book Cole [4].

Note 4.1 In view of result 2.5, the non-linear system (1.3) is equivalent to

$$x' + (K_1 \otimes K_2)x = h(t, x) \qquad (4.2)$$

where

$$h(t, x(t)) = (A^+ \otimes B^+)g + [I_{nq} - (A^+A \otimes B^+B)(t)]v,$$

v is a nq-column vector.

Lemma 4.1 The fundamental matrix $X(t)$ of the homogeneous equation (2.7), then the solution of (4.2) is given by

$$x = XL + X\int_a^t X^{-1}(s)\ h(s, x(s))\ ds$$

(4.3)

where L is a constant nq-column vector.

Lemma 4.2 The fundamental matrix $X(t)$ of the homogeneous equation (2.7), then the vector $\Psi(t)$ defined by $\Psi(t)=X(t)E^+\beta$ is also a solution for the equation (2.7).

Theorem 4.1 Suppose that the homogeneous boundary value problem is in compatible and g(t, x(t)) satisfies Lipschitz condition (4.1), then if

$$MN \max_{t\in[a,c]} \int_a^c |G_1(t,s)ds| < 1$$

(4.4)

the problem(4.2) satisfying(1.2) has one and only one solution, where $G_1(t,\ s)$ is the Green's matrix for the corresponding homogeneous BVP and

$$N = \max_{t\in[a,c]} \left|(A\otimes B)^+\right|.$$

Proof: From lemma 4.1 the solution of (4.2) is given by (4.3) and $X(t), L$ are defined earlier also must satisfy the B.Cs (1.2). Substituting the general form of x(t) in the boundary conditions(1.2), we have

$$[(L_1 \otimes M_1)X(a) + (L_2 \otimes M_2)X(b) + (L_3 \otimes M_3)X(c)]L$$

$$+ (L_2 \otimes M_2)X(b)$$

$$\times \int_a^b X^{-1}(s)h(s, x(s))ds + (L_3 \otimes M_3)X(c)$$

$$\times \int_a^c X^{-1}(s)h(s, x(s))ds = \beta.$$

$$EL = \beta - (L_2 \otimes M_2)X(b)$$

$$\times \int_a^b X^{-1}(s)h(s, x(s))ds - (L_3 \otimes M_3)X(c)$$

$$\times \int_a^c X^{-1}(s)h(s, x(s))ds = \eta.\quad (4.5)$$

Suppose the equation (4.5) in 'L' is consistent, then $(I_{nq}-E^+E)=O$ and the unique least square solution 'L' is given by L=$E^+\eta$. Substituting the value of 'L' in the solution of x(t) in the boundary conditions(1.2), we get

$$x = XE^+\eta + X\int_a^c G_1(t,s)\ h(s,x)\ ds$$

where

$$G_1(t,s)\Big|_{t\in[a,b]} = \begin{cases} X(t)E^+(L_1 \otimes M_1)X(a)X^{-1}(s), \\ \qquad a < s < t \le b < c \\ -X(t)E^+[(L_2 \otimes M_2)X(b) \\ \qquad + (L_3 \otimes M_3)X(c)]X^{-1}(s), \\ \qquad a \le t < s < b < c \\ -X(t)E^+(L_3 \otimes M_3)X(c)X^{-1}(s), \\ \qquad a \le t < b < s < c \end{cases}$$

$$G_1(t,s)\Big|_{t\in[b,c]} = \begin{cases} X(t)X^{-1}(s) \\ \quad -X(t)E^+(L_3 \otimes M_3)X(c)X^{-1}(s), \\ \qquad a < b < s < t \le c \\ -X(t)E^+(L_3 \otimes M_3)X(c)X^{-1}(s), \\ \qquad a < b \le t < s < c \\ X(t)E^+(L_1 \otimes M_1)X(a)X^{-1}(s), \\ \qquad a < s < b < t \le c \end{cases}$$

Let $X(t)E^+\beta = \Psi(t)$. The linear space 'J' of functions $x \in C^{nq}[a,c]$ with the norm $\|x(t)\| = \max_{t\in[a,c]}|x(t)|$. Hence the linear space 'J' is a Banach space.

Now define the mapping $T: J \rightarrow J$ is

$$Tx = \int_a^c G_1(t,s)\ h(s,x)\ ds$$

Then, if $u, w \in J$

$$\left| Tu(t) - Tw(t) \right| = \left| \int_a^c G_1(t,s)h(s,u(s))ds - \int_a^c G_1(t,s)h(s,w(s))ds \right|$$

$$= \left| \int_a^c G_1(t,s)[h(s,u(s)) - h(s,w(s))]ds \right|$$

$$\leq \int_a^c \left| G_1(t,s) \right| \left| (A \otimes B)^+(s) \right| \left| g(s,u(s)) - g(s,w(s)) \right| ds$$

$$\leq M \left| (A \otimes B)^+ \right| \int_a^c \left| G_1(t,s) \right| \left| u(s) - w(s) \right| ds.$$

Hence

$$\left\| Tu - Tw \right\| \leq MN \left\| u - w \right\| \max_{t \in [a,c]} \int_a^c \left| G_1(t,s) \right| ds.$$

$$\left\| Tu - Tw \right\| \leq \zeta \left\| u - w \right\|,$$

where

$$\zeta = MN \max_{t \in [a,c]} \int_a^c \left| G_1(t,s) \right| ds < 1.$$

Since $\zeta < 1$, the mapping is contraction and hence by Banach fixed point theorem $\exists$ a unique solution to the BVP (4.2) with boundary condition

$$(L_1 \otimes M_1)x(a) + (L_2 \otimes M_2)x(b) + (L_3 \otimes M_3)x(c) = 0. \quad (4.6)$$

By applying the above procedure to the BVP

$$x' + (K_1 \otimes K_2)x = h(t, x + \psi), \quad (4.7)$$

and(4.6) a unique solution x_1 is constructed. Define $z = x_1 + \Psi$, then z is a solution of (4.7) satisfying (4.6). Hence z is a unique solution to the BVP (4.2) with boundary condition (1.2).

Remark 4.1 Here $G_1(t, s)$ satisfies the usual properties of Green's matrix.

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